APPROXIMATION BY BERNSTEIN TYPE POLYNOMIALS

DISSERTATION

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(Ariz Khan)

PREFACE

The theory of approximation of functions is an extremely versatile and variegated one. It has a close relationship with other branches of mathematics. Existence of such a relationship is explained by the fact that many important problems of approximation theory are formulated and solve in the process of development of other mathematical topics, while on the other hand, the development of approximation theory has assisted development in other mathematical domains and set the course of completely new directions in mathematics. The approximation theory is a branch of mathematical analysis engaged in problems of approximate representation of arbitrary functions with aid of simplest analytical devices.

Sergi Natanovich Bernstein worked on the theory of best approximation of functions. He greatly extended work begun by chebyshev in 1854. In 1911 he introduced what are now called the **Bernstein Polynomials** to give a constructive proof of **Weierstrass theorem (1885)**, namely that a continuous function on a finite subinterval of the real line can be uniformly approximated as closely as we wish by a polynomial.

At the international congress at Cambridge in 1912, **Bernstein** talked about this work. He than continued to develop these ideas, solving problems in interpolation theory, methods of mechanical integration and, in 1914, introduced a new class of quasi-analytic functions.

Some of **Bernstein's** most important work was in the theory of probability. He attempted an axiomatisation of probability theory in 1917. He generalized Lyapunov's conditions for the central limit theorem, studied generalization of the law of large numbers, worked on Markov process and stochastic process. **Bernstein** also studied application of probability.

The Present Dissertation comprises fifth chapters and each chapter consists of various sections which are numbered in order

In chapter 1, I have given a resume of previous work that is closely related to other chapters.

Chapter 2, deals with the approximation of functions by Bernstein polynomial in which, I have given several theorems on the approximation of functions by Bernstein polynomial, proved by **S.N. Bernstein, popovic and Lu-wincin.**

Chapter 3, deals with the approximation of functions by generalized Bernstein polynomial in which I have given some theorems about the approximation by generalized Bernstein polynomial which shows that the generalized Bernstein Polynomial approximate the function f(x) more closely to Bernstein polynomial.

Chapter 4, concerned with Lipschitz constant for Bernstein polynomial and also have concerned Lipschitz constant on a rectangle.

Chapter 5, deals with Bernstein polynomial on a simplex in which also tested linear combination of derivatives and smoothness of function and asymptotic behaviour of Bernstein polynomial.

In the last, I have given a complete bibliography on this topic.

CONTENT

CHAPTER-1

PRELIMINARIES

CHAPTER - 1

PRELIMINARIES

1.0. INTRODUCTION:

In approximation theory of real function, one of the most important theorems is the well known theorem of Weierstrass. It states that every continuous function on a compact interval, $f:[0,1] \to R$, is the limit of the sequence of polynomials uniformly converging towards this function. One of the most elegant proofs of the Weierstrass theorem was given by Bernstein [3] in 1912, it is based upon a consideration drawn from probability theory. In the proof, a sequence of polynomials is constructed, now bearing his name, which depend on the function to be approximated and if their order increases infinitely they tend uniformly to that function. This fact amplified and enriched the researches in this domain with new and important results.

1.1 LINEAR POSITIVE OPERATOR:

In 1953 Korovkin P.P. [14] introduces the concept of linear positive operator in the approximation theory although the corresponding idea already occurred in 1952 in a paper of Bohman.

Definition 1.1.1. Linear Operator:

An operator $L(f;x)$ mapping C^* into C^* is said to be "linear" if the domain of its existence together with its functions $f(t)$ and ϕ (t), contains the function $af(t) + b\phi(t)$, then

$$L(a f + b \phi ; x) = a L(f ; x) + b L (\varphi ; x)$$

where a, b are any real numbers.

Definition 1.1.2. Linear Positive Operator:

Let L be a linear operator mapping C^* into C^* (on C[a,b] into C[c,d] with $[c, d] \subseteq [a,b]$). We say L is positive if for each non-negative function f in C^* , the resulting function L(f) is also non-negative.

Remarks: It follows from the definition of linear operators that for any two functions f and g in C^*, if $f \leq g$, then

$$L(f) \leq L(g)$$

1.2 APPROXIMATION:

The approximation problem can be described in the following way (see G.G. Lorentz [16]).

Let ϕ be a set of functions defined on A. If f is a function on a space A. Can one find a linear combination.

$$P = a_1 \phi_1 + a_2 \phi_2 + \text{-----} + a_n \phi_n \qquad \text{---1.2.1}$$

where $\phi_i \in \Phi$ and a_i are reals which is close to f.

Definition 1.2.1:

Let X be a Banach space of continuous functions on [a,b] with the norm $||.||$ defined by

$$||f|| \le \underset{x\varepsilon[a,b]}{Sup.}|f(x)|$$

Let ϕ be a subset of X, an element f on X is called approximable by linear combination

$$P = a_1 \phi_1 + a_2 \phi_2 + \text{-----} + a_n \phi_n,\ \phi_i \in \Phi$$

and a_i are real, if for each $\in > 0$, there is a polynomial P such that

$$||f - P|| < \in$$

Definition 1.2.2. Degree of Approximation:

Let $\phi = \{\phi_i\}$ be a sequence of functions, then

$$E_n^* (f) = E_n (f) = \inf_{a_1, a_2, \ldots, a_n}. \ ||f - (a_1\phi_1 + a_2\phi_2 + \text{---} + a_n \phi_n)||$$

$$= \inf_{a_1, a_2, \ldots, a_n}. \ ||f - P|| \qquad \text{------}1.2.2.$$

Where, P is as defined in (1.2.1), is called the n^{th} degree of approximation of f by the sequence $\{\phi_i\}$.

Definition 1.2.3. Best Approximation:

If the infimum in (1.2.2) attains for some P, then P is called a linear combination of best approximation or polynomial of best approximation.

Remarks: (i) If P are algebraic or trigonometric polynomials of given degree, then n in (1.2.2) will refer to the degree of the polynomials rather than to the number of functions ϕ_i.

(ii) $E_n(f)$ is also called the error in approximating f by the polynomial P.

The basis of the theory of approximation of functions of a real variable is a theorem due to Weierstrass which states as:

Weierstrass Theorem 1.2.1:

For each function f(x), continuous on a closed interval [a,b] and for each $\in > 0$, there is a polynomial P(x) approximating f(x) uniformly with an error less than $\in$. i.e.

$$|f(x) - P(x)| < \in$$

1.3. CLASSES OF FUNCTIONS:

In this section, we list several classes of functions. Let $f(x) \in C[a,b]$ (or $f(x) \in C_{2\pi}$) and let E_n be its best approximation by means of algebraic (or trigonometric) polynomials of order not higher than n. By Weierstrass's theorem, it is found that

$$\underset{n\to\infty}{\lim it}\ E_n = 0$$

Naturally, the "simpler" the approximating function f(x), the more accurately will it be represented by means of a polynomial (algebraic or trigonometric).

Now, we shall engage in the question of the influence exerted by an improvement in the structural properties of the approximated function on the order of the decrease of its best approximation E_n.

A convenient characteristic of the structural properties of a function is a quantity called the "modulus of continuity" of this function.

Definition 1.3.1. Modulus of Continuity:

The degree of approximation of function f(x), $a \leq x \leq b$ by polynomials may be simply described in terms of modulus of continuity.

For each $\delta>0$, the modulus of continuity $\omega(\delta)$ is the maximum of $|f(x) - f(y)|$ for all $a \leq x \leq b$, $a \leq y \leq b$, $|x-y|<\delta$,

$$\omega(f; \delta) = \omega(\delta) = \max.\{|f(x) - f(y)|, x, y \in [a, b], |x-y|<\delta\}$$

The modulus of continuity has the following fundamental properties.

(a) $\omega(\delta) \to 0$ as $\delta \to 0$

(b) $\omega(\delta) \geq 0$ and non-decreasing

(c) ω is sub-additive

$$\omega(\delta_1+\delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$$

(d) $\omega(\delta)$ is continuous

(e) $\omega(n\delta) \leq n\,\omega(\delta)$

(f) $\omega(\lambda\delta) \leq (\lambda + 1)\,\omega(\delta)$, for $\lambda \geq 0$

Remark: Since by linear combination, the interval [a,b] may be transformed into [0,1] and so modulus of continuity may be defined in the closed interval [0,1] as

$$\omega(\delta) = \max\{|f(x) - f(y)| : x, y \,\varepsilon [0,1], |x-y|<\delta\}$$

Definition 1.3.2 Lipschitz Class Lipα:

Suppose $0 < \alpha \leq 1$. the Lipschitz Class Lipα is the space of all functions f satisfying the condition

$$||f||_\alpha = \underset{t>0,x}{Sup.}\left(\frac{|f(x+t)-f(x)|}{t^\alpha}\right) < \infty \qquad \text{------- } 1.3.2.$$

Definition 1.3.3. Lip. K(t) class:

A function f(x), integrable L(Lebesgue integrable) is said to belong to Lip K(t) class if

$$||f||_K = \underset{t>0,x}{Sup.}\left(\frac{|f(x+t)-f(x)|}{k(t)}\right) < \infty$$

Where k(t) is positive increasing function, such that

$\frac{k(t)}{t}$ is decreasing, $K(t) \to 0$

as $t \to 0$, and

$K(x,y) \leq K(x)\, K(y)$

Definition 1.3.4: If the function f(x) defined on the interval [a,b] and for all x, y $\in$[a,b], it satisfies the inequality.

$$|f(y) - f(x)| \leq M\,|y-x|^\alpha \qquad \text{--------}1.3.4$$

then it is said that the function f(x) satisfies the Lipschitz condition with exponent α and coefficient M, and we write

$$f(x) \in Lip_M \alpha$$

In those cases, when the coefficient M is not essential then we write $f(x) \in Lip.\ \alpha$

In other words, $Lip_M\alpha$ is the class of all the functions satisfying the Lipschitz condition of a given order with special coefficient M and Lip α is the class of functions satisfying the Lipschitz condition of order α with arbitrary coefficient.

A function f(x) satisfying the Lipschitz condition has the following properties:

(a) A function satisfying the Lipschitz condition is uniformly continuous.

(b) If $f(x) \in Lip\alpha$, where $\alpha > 1$, then f(x) is a constant quantity.

(c) If everywhere in the interval (a, b), there exists a derivatives

$f'(x)$ with $|f'(x)| \leq M$, then

$f(x) \in Lip_M 1$

(d) If the interval [a,b] is finite and $\alpha < \beta$, then Lip. $\alpha \supset$Lip. β

(e) The relations $f(x) \in Lip_M\alpha$ and $\omega(h) \leq Mh^{\alpha}$ are equivalent.

Definition 1.3.5. Hardy – Little Wood Class Lip(α, p):

The Hardy-Little wood class Lip(α, p) ($p<\infty$), is the space of all functions f satisfying the condition.

$$\|f\|_{\alpha,p} = \sup_{t>0,x}.\left(\int_0^{2\pi}\left|\frac{f(x+t)-f(x)}{t^{\alpha}}\right|^p dt\right)^{\frac{1}{p}} < \infty$$

1.4. CERTAIN POLYNOMIALS:

Bernstein Polynomials 1.4.1: For a function f(x) defined on the closed interval [0,1]. the expression

$$B_n(x) = B_n(f\,;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \qquad \text{-----}1.4.1$$

is called the Bernstein polynomial of order n of the function f(x). B_n (x) is a polynomial in x of degree $\leq$ n.

Remarks:

(i) The expression

$$P_{n,K}(x) = \binom{n}{k} x^k(1-x)^{n-k}$$

Used in (1.4.1) are the binomial or Newton probabilities well known in the theory of probability.

(ii) B_n (x) is linear with respect to the function f(x).

$$B_n(f;x) = a_1B_n(f_1;x) + a_2B_n(f_2;x)$$

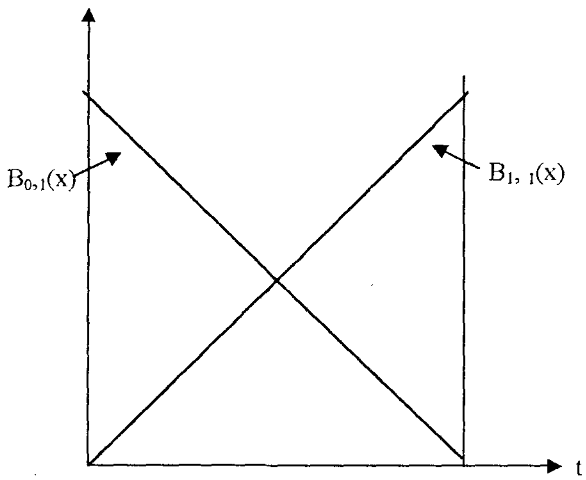
$$\text{if, } f = a_1f_1 + a_2f_2$$

Corollary 1.4.1(a): The Bernstein polynomials of degree 1 are

$$B_{0,1}(x) = 1 - x$$

$$B_{1,1}(x) = x$$

and can be plotted for $0 \leq x \leq 1$ as



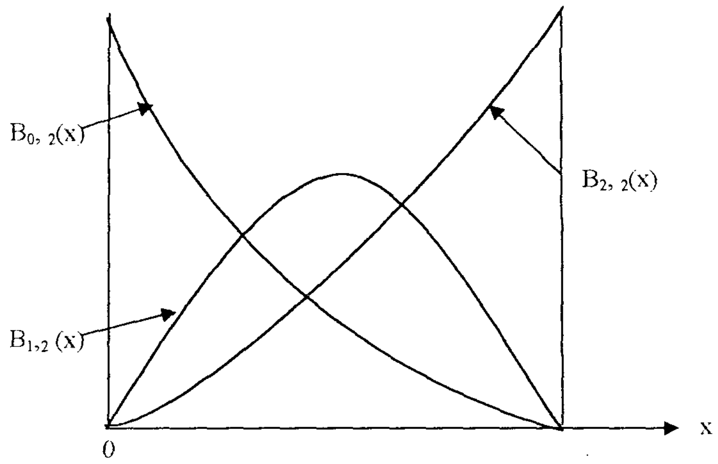
(b) The Bernstein polynomials of degree 2 are

$$B_{0,2}(x) = (1-x)^2$$

$$B_{1,2}(x) = 2x\,(1-x)$$

$$B_{2,2}(x) = x^2$$

and can be plotted for $0 \leq x \leq 1$ as



Derivative of Bernstein Polynomial 1.4.2:

Derivative of the n^{th} degree Bernstein polynomials are polynomials of degree n-1. This derivative can be written as a linear combination of Bernstein polynomials. In particular

$$\frac{d}{dx}B_{k,n}(x) = \frac{d}{dx}\binom{n}{k}x^k(1-x)^{n-k}$$

$$= \frac{k\,n!}{k!(n-k)!}x^{k-1}(1-x)^{n-k} + \frac{(n-k)n!}{k!(n-k)!}x^k(1-x)^{n-k-1}$$

$$= \frac{n(n-1)!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k} + \frac{n(n-1)!}{k!(n-k-1)!}x^k(1-x)^{n-k-1}$$

$$= n\left(\frac{(n-1)!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k} + \frac{(n-1)!}{k!(n-k-1)!}x^k(1-x)^{n-k-1}\right)$$

$$= n(B_{k-1,n-1}(x) - B_{k,n-1}(x))$$

for $0 \leq k \leq n$

That is, the derivative of Bernstein Polynomial can be expressed as the degree of the polynomial, multiplied by the difference of two Bernstein polynomials of degree n-1.

Bernstein Polynomial on an unbounded Interval 1.4.3:

Let the function f(x) be defined on the interval (o,b), b>0.

To obtain the Bernstein polynomials. B_n (f ;x, b) for this interval, we make the substitution $y = x\, b^{-1}$ in the polynomial $B_n(\phi, y)$ of the function $\phi(y) = f(b.y)$, $0 \leq y \leq 1$ and obtain in this way:

$$B_n\ (x;\ b) = B_n\ (f;\ x,\ b) = \sum_{k=0}^{n} f\left(\frac{bk}{n}\right) p_{n.k}(x/b)$$

Where,

$$p_{n,k}(x/b) = \binom{n}{k}\left(\frac{x}{b}\right)x^{k}\left(1-\frac{x}{b}\right)^{n-k}$$

Schurer's Polynomial 1.4.4:

Schurer, F. [26] in his report defined an analogous operator

$$B_{nr}\ :\ c\left[0,1+\frac{r}{n}\right] \to c[0,1]$$

In such a way that

$$B_{nr}\ (f;\ x) = \sum_{k=0}^{n+r} f\left(\frac{k}{n}\right) P_{nr,k}(x)$$

Where

$$P_{nr,k}(x) = \binom{n+r}{k} x^k (1-x)^{n+r-k}$$

and r is a non-negative integer

In case r = 0, it reduces to Bernstein Polynomials (1.4.1)

Modified Bernstein Polynomial 1.4.5:

Let f(x) is integrable on [0,1] and consider the definite integral.

$$F(x) = \int_0^x f(t)dt$$

of f(x) and calculate the derivative of the polynomial $B_n(F;x)$..

Replacing B_n by B_{n+1} we obtain. (see G.G. Lorentz [15])

$$P_n(x) = P_n(f;x) = \frac{d}{dx} B_n(F;x)$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (n+1) \int_{K/(n+1)}^{(k+1)/(n+1)} f(t)\, dt$$

which approximate any integrable function f(x) and called modified Bernstein polynomial.

Trigonometric Polynomial 1.4.5:

The function

$$T_n(x) = a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$$
$$+ \text{-------------} + (a_n \cos nx + b_n \sin nx)$$

is called a trigonometric polynomial of order n, if $a_n^2 + b_n^2 \neq 0$ and the series

$$\frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x)$$
$$+ \text{-------------} + (a_n \cos nx + b_n \sin nx) + \text{-----------}$$

is called a trigonometric series

1.5 CAUCHY'S INEQUALITY:

Definition 1.5.1:

$$|\Sigma\, a_n\, b_n| \leq (\Sigma|a_n|^2)^{1/2}\,(\Sigma|b_n|^2)^{1/2}$$

1.6. ORDER OF SATURATION:

The most important and recent concept of approximation theory is 'Saturation'. Many of the classical approximation procedure such as Fourier Series, Fejer and Jackson means the Bernstein polynomials have properties that rate at which they converge, is dependent on the smoothness of the function being approximated.

Furthermore, it often happens that there exists an 'optimal' order of approximation in the sense that a better rate of approximation can not be achieved by increasing the smoothness of the function. When this occurs, we say the approximation procedure is saturated and the saturation class is the collection of all functions optimally approximated by the procedure.

The concept of saturation class was first introduced into approximation theory by J. Favard in 1949.

Definition (1.6.1.) Saturation:

Let f(x) be an integrable function in $(-\pi,\pi)$ and periodic with period 2π and let its Fourier series be

$$S(f) = \tfrac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

$$= \sum_{k=0}^{\infty} A_k(x)$$

Let us consider the family of linear operators.

$$L_n(f;x) = \sum_{k=0}^{\infty} g_k^{(n)} A_k(x)$$

Where,

$$g_k^{(n)}\ ,\quad k=0,1,2\text{--------}(\,g_0^{(n)} = 1\,)$$

are summating functions.

If there is a positive non-increasing function $\phi(x)$ and a constant k of functions such that

$$\|f(x) - L_n(f;x)\| = 0(\phi(x))$$

if and only if f(x) is constant, and

$$\|f(x) - L_n(f;x)\| = 0(\phi(x))$$

if and only if f(x) belongs to class k; then it is said to be that their method of approximation is saturated with the order $\phi(x)$ and the class k. The function ϕ is then called an optimal degree of approximation.

1.7. APPLICATION:

In applications, a matrix formulation for the Bernstein polynomials is useful. These are straight forward to develop if one only looks at a linear combination in terms of dot products.

Given a polynomial written as a linear combination of the Bernstein basis functions

$$B(x) = C_0 B_{0,n}(x) + C_1 B_{1,n}(x) + \text{----------} + C_n B_{n,n}(x)$$

It is easy to write this as a dot product of two vectors

$$B(x) = [B_{0,n}(x) \quad B_{1,n}(x) \text{ ----------- } B_{n,n}(x)] \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ | \\ C_n \end{bmatrix}$$

We can convert it as :

$$B(x) = [1 \ x \ x^2 \text{ ---------- } x^n] \begin{bmatrix} b_{0,0} & 0 & 0 \text{ ---------0------------ } 0 \\ b_{1,0} & b_{1,1} & 0 \text{ ---------0------------ } 0 \\ b_{2,0} & b_{2,1} & b_{2,2} \text{ ---------0------------ } 0 \\ | & | & | \qquad\qquad\qquad\qquad | \\ b_{n,0} & b_{n,1} & b_{n,2} \text{ ---------------------- } b_{n,n} \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ | \\ C_n \end{bmatrix}$$

Where the $b_{i,j}$ are the coefficients of the power basis that are used to determine the respective Bernstein polynomials. We note that the matrix in this case is lower triangular.

Corollary(a): In the quadratic case (n=2). The matrix representation is

$$B(x) = [\ 1 \ x \ x^2] \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix}$$

(b) In the cubic case (n=3), the matrix representation is

$$B(x) = [1 \ x \ x^2 \ x^3] \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{bmatrix}$$

CHAPTER-2

THE DEGREE OF APPROXIMATION OF FUNCTION BY BERNSTEIN POLYNOMIALS

CHAPTER – 2

THE DEGREE OF APPROXIMATION OF FUNCTION BY BERNSTEIN POLYNOMIALS

2.1. INTRODUCTION:

Let f(x) be a function defined on the segment [0,1]. If n is a positive integer and k is an integer such that $0 \le k \le n$.

The binomial coefficient $\binom{n}{k}$ defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The polynomial

$$B_n(f;x) = B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right) \qquad \text{-----}2.1.1$$

is called the Bernstein Polynomial of order n of the function f(x). Clearly this is a positive linear operator which maps C(0,1) into itself, by the binomial formula.

$$\left.\begin{aligned} B_n(e,x) &= \sum_{k=0}^{n} p_{nk}(x) = 1 \\ \text{where} \quad p_{n,k}(x) &= \binom{n}{k} x^k (1-x)^{n-k} \end{aligned}\right\} \qquad \text{---------} 2.1.2$$

and hence $\|B_n\| = 1$, for n =0,1,2, -------------

If we differentiate (2.1.2) with respect to x, then we get

$$\sum_{k=0}^{n} \binom{n}{k} \left[kx^{k-1}(1-x)^{n-k} - (n-k)x^k(1-x)^{n-k-1} \right] = 0$$

or

$$\sum_{k=0}^{n}\binom{n}{k}x^{k-1}(1-x)^{n-k-1}(k-nx)=0$$

on multiply both side by x(1-x), we have

$$\sum_{k=0}^{n}\binom{n}{k}x^{k}(1-x)^{n-k}(k-nx)=0 \qquad \text{--------} \; 2.1.3$$

or

$$\left.\begin{aligned}&\sum_{k=0}^{n}p_{n.k}(x)(k-nx)=0\\ &\text{or}\\ &\sum_{k=0}^{n}kp_{n,k}(x)=nx\sum_{k=0}^{n}p_{nk}(x)=nx\end{aligned}\right\} \qquad \text{-----------} \; 2.1.4$$

on differentiating (2.1.3) with respect to x, we get

$$\sum_{k=0}^{n}\binom{n}{k}\left[-nx^{k}(1-x)^{n-k}+x^{k-1}(1-x)^{n-k-1}(k-nx)^{2}\right]=0 \; \text{------} 2.1.5$$

Apply (2.1.2) to (2.1.5), gives

$$\sum_{k=0}^{n}\binom{n}{k}x^{k-1}(1-x)^{n-k-1}(k-nx)^{2}=n$$

or

$$\sum_{k=0}^{n}\binom{n}{k}x^{k}(1-x)^{n-k}(k-nx)^{2}=nx(1-x) \qquad \text{------} \; 2.1.5\ (A)$$

or

$$\sum_{k=0}^{n} p_{n,k}(x)(k^2 - 2nkx + n^2x^2) = nx(1-x)$$

or

$$\sum_{k=0}^{n} k^2 p_{n,k} \; - 2nx\sum_{k=0}^{n} k \; p_{n,k} + n^2x^2\sum_{k=0}^{n} p_{n,k} = nx(1-x)$$

Now from (2.1.4), we have

$$\sum_{k=0}^{n} k^2 p_{n,k} - 2nx.nx + n^2x^2.1 = nx(1-x)$$

or

$$\sum_{k=0}^{n} k^2 p_{n,k} = n^2x^2 + nx(1-x) \qquad \text{-------- 2.1.6}$$

useful in connection with Bernstein Polynomials are the sums

$$T_{n,r}(x) = \sum_{k=0}^{n} (k-nx)^r p_{n,k}(x), \quad r = 0, 1, \ldots\ldots\ldots\ldots \qquad \text{------2.1.7}$$

then,

$$T_{n,0}(x) = \sum_{k=0}^{n} p_{n,k}(x) = 1 \qquad \text{(from 2.1.2)}$$

$$T_{n,1}(x) = \sum_{k=0}^{n} (k-nx)p_{n,k}(x)$$

$$= \sum_{k=0}^{n} k\, p_{n,k}(x) - nx\sum_{k=0}^{n} p_{n,k}(x)$$

= nx- nx (from (2.1.2) & (2.1.4)

= 0

and

$$T_{n,2}(x)=\sum_{k=0}^{n}(k-nx)^2 p_{n,k}(x)$$

$$=\sum_{k=0}^{n}(k^2-2knx+n^2x^2)p_{n,k}(x)$$

$$=\sum_{k=0}^{n}k^2 p_{n,k}(x)-2nx\sum_{k=0}^{n}kp_{n,k}(x)+n^2x^2\sum_{k=0}^{n}p_{n,k}(x)$$

$= n^2x^2 + nx(1-x) -2nx.\ nx +n^2x^2$

[from (2.1.6), (2.1.4) & (2.1.2)]

$= nx(1-x)$

In order to find the value $T_{n,r}$ for $r \geq 2$, it is convenient to use the recurrence relation

$$T_{n,r+1} = x(1-x)\,[T'_{n,r} + nr\, T_{n,\, r-1}], \qquad r \geq 1 \qquad \text{----- } 2.1.8$$

which follows from the fact that

$$\frac{d}{dx}p_{n,k}(x)=\binom{n}{k}\left[kx^{k-1}(1-x)^{n-k}-(n-k)(1-x)^{n-k-1}x^k\right]$$

$$=\frac{1}{x(1-x)}p_{n,k}[k(1-x)-(n-k)x]$$

$$= \frac{k - nx}{x(1-x)} p_{n,k}(x)$$

and from the differentiation of (2.1.7) with respect to x. Thus we obtained from (2.1.7) and (2.1.8)

$T_{n,0} = 1, \quad T_{n,1} = 0, \quad T_{n,2} = nX$

$$\begin{aligned} T_{n,3} &= X(T'_{n,2} + 2n\, T_{n,1}) \\ &= X(n(1-2x) + 0) \\ &= n(1-2x)\, X \\ T_{n,4} &= x(1-x)\, [T'_{n,3} + 3nT_{n,2}] \\ &= X[n(1-2x)^2 - 2nx(1-x) + 3n^2 X] \\ &= 3n^2X^2 - 2nX^2 + nX(1-2x)^2 \end{aligned}$$

where $X = x(1-x)$

It is easy to see that f(x) is continuous, then at large value of n Bernstein polynomial differs very little from f(x). In fact, we have already seen that in the sum $\sum_{k=0}^{n} p_{n,k}(x)$, those terms for k/n is different from x play no roll at all. This holds also for the polynomial B_n (f;x). since the factors f(k/n) are bounded. Therefore, the only terms of essential importance in the polynomial $B_n(x)$ are those for which k/n is very closed to x. But for such terms the factor f(k/n) hardly differs from f(x). This means that the polynomial $B_n(x)$ remains almost unchanged if f(k/n) is replaced in its terms by f(x). In other words, the following approximate equality holds,

$$B_n(f;x) \approx \sum_{k=0}^{n} f(x)\binom{n}{k}x^k(1-x)^{n-k}$$

$$\approx f(x) \text{ from } (2.1.2)$$

The following theorem gives an exact formulation of this leading reasoning:

Theorem2.1: (S.N. Bernstein[3]):

If f(x) is a continuous function on [0,1], then

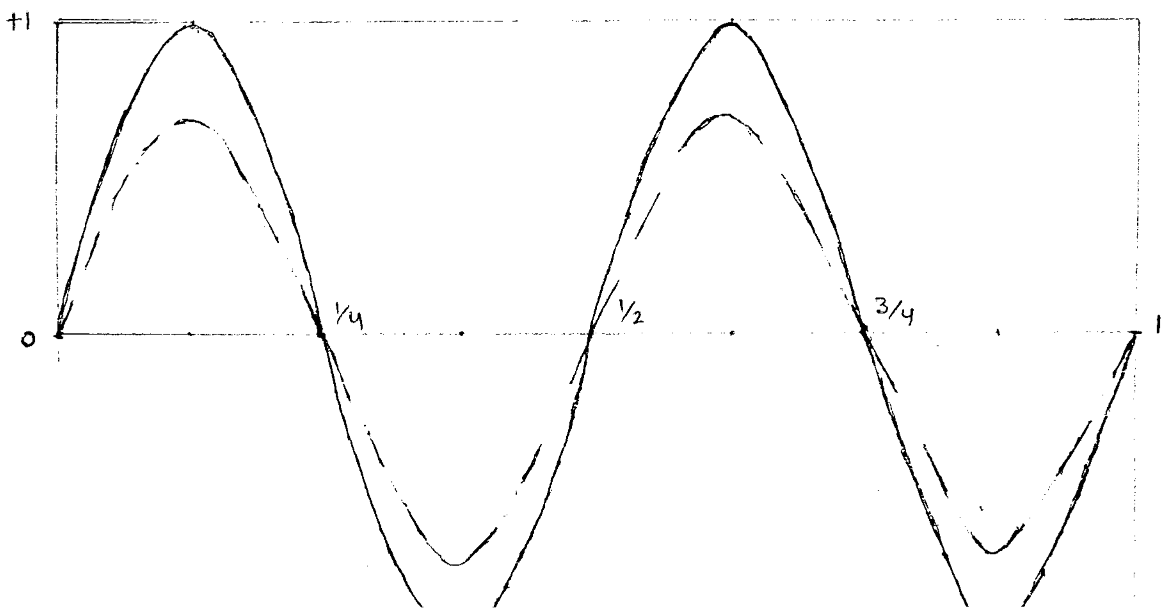
$$\lim_{n\to\infty} B_n(f;x) = f(x)$$

holds, uniformly with respect to x, $0 \leq x \leq 1$

Proof – See ([16], on page 9-10)

Here, is a graphical example of Bernstein polynomial converging to their limit.

Bernstein Polynomial approximations converging to sin(4πx) on [0,1]



2.2. APPROXIMATION OF CONTINUOUS FUNCTION BY BERNSTEIN POLYNOMIAL:

The degree of approximation of a function f(x), $a \leq x \leq b$ by polynomials may be simply described in terms of its modulus of continuity $\omega(\delta) = \omega(f;\delta)$.

The degree of approximation of a function by arbitrary polynomials is given by the theorems of D. Jackson(see Jackson[11].

Theorem(2.2.1): If $\omega(\delta)$ is the modulus of continuity of f(x), $0 \leq x \leq 1$, the for each n, there is a polynomial $P_n(x)$ of degree $\leq n$, such that

$$|f(x) - P_n(x)| \leq C\ \omega(n^{-1})$$

where C is an absolute constant

Theorem(2.2.2): If f(x) has a continuous p^{th} derivative $f^{(p)}(x)$ in [0,1] with modulus of continuity $\omega_p(\delta)$ then for each n>p, there is a polynomial $P_n(x)$ of degree $\leq$ n for which

$$|f(x) - P_n(x)| \leq C_p n^{-p} \omega_p(n^{-1})$$

C_p is again an absolute constant

We shall see that the approximation is generally not as good as in the case of arbitrary polynomials $P_n(x)$

Now we have the following estimate for the goodness of the approximation.

Theorem 2.2.3: (T. Popoviciu)[21])

If f(x) is continuous and $\omega(\delta)$ is the modulus of continuity of f(x), then

$$|f(x) - B_n(x)| \le 5/4\ \omega\ (n^{-1/2})$$

holds for an arbitrary natural number n,

Lu, Wen-cin[17] established the more precise estimate

$$|f(x) - B_n(x)| \le \frac{19}{16}\omega\left(\frac{1}{n}\right)$$

Later, Moldovan [19] established the more precise estimation comparison of the estimate of Lu, Wen-cin.

Theorem (2.2.4): Let f(x) $\in c[0,1]$, $B_n(f;x)$ be its Bernstein Polynomial, $\omega(h)$ be the modulus of continuity of f(x), then the inequality.

$$|f(x) - B_n(f;x)| \le (1+S)\ \omega\left(\frac{1}{n}\right) \qquad \text{---------2.2.4}$$

with S=0.093785 holds for an arbitrary natural no. n

Later P.C. Sikhema [24] has proved the following estimate which is more precise estimation in comparison of (2.2.4)

Theorem(2.2.5): Let f(x) be real valued and continuous function on [0,1]. Let $\omega(\delta)$ be its modulus of continuity. Let $B_n(f;x)$ be its Bernstein Polynomial then

$$\underset{0 \le x \le 1}{Max.}\ |f(x) - B_n(f;x)| \le K\omega\left(\frac{1}{n}\right)$$

When $K = \frac{4306 + 837\Gamma 6}{5832} = 1.0896776$

holds for an arbitrary natural no. n.

G.M. Mirakjan [18] established a general expression of the approximation of function by Bernstein Polynomial.

Theorem (2.2.6): Let f(x) $\in C[0,1]$, $\omega(\delta)$ be its modulus of continuity, and $B_n(f;x)$ be its Bernstein Polynomial then

$$\underset{0\le x\le 1}{Max.} \left|f(x) - B_n(f;x)\right| \le \left[1 + \frac{1}{\sqrt{\pi}} \frac{\frac{\Gamma\alpha+1}{2}}{2^{\alpha/2}}\right] \omega\left(\frac{1}{\sqrt{n}}\right)$$

holds for arbitrary $\alpha \ge 1$

Corollary (2.2.7): If $(x) \in Lip_m \alpha$ and $B_n(f;x)$ be its Bernstein polynomial, then

$$|f(x) - B_n(f;x)| \le 3/2 \ \frac{M}{\Gamma n^{\alpha}}$$

2.3. APPROXIMATION OF FUNCTION WHICH HAVE CONTINUOUS DERIVATION ON [0,1] BY BERNSTEIN POLYNOMIAL:

Now we discuss the approximation of the function f(x) which have a continuous derivative f'(x) on [0,1]

Theorem (2.3.1): If $\omega_1(\delta)$ is the modulus of continuity of $f'(x)$ and $B_n(f;x)$ be the Bernstein polynomial of the function f(x) defined on [0,1], then

$$|f(x) - B_n(f;x)| \leq \tfrac{3}{4}\, n^{-1/2}\, \omega_1(n^{-1/2}) \qquad \text{-----2.3.1.1}$$

Corollary: In particular, if $f'(x) \in Lip(1)$, then

$$|f(x) - B_n(f;x)| \leq c(1/n) \qquad \text{------2.3.1.2}$$

or

$O(1/n)$

holds uniformly in x, $0 \leq x \leq 1$

Furthermore, this last estimate $O(1/n)$ can not be improved by assuming the existence of the second or higher derivative of f(x), and then there is problem that

$$|f(x) - B_n(f;x)| = 0(1/n) \qquad \text{------2.3.1.3}$$

can never be true, except if f(x) is a linear function.

Therefore, it is conjecture that the equality (2.3.1.3) is only true when f(x) is a linear function.

Theorem (2.3.2): Let f(x) be bounded in [0,1], and has a finite second derivative f"(x) at a certain point $x_0 \in [0,1]$ then E.V. Voronovskaya shows that

$$B_n(f;x_0) = f(x_0) + \frac{f''(x_0)}{2n} x_0(1-x_0) + \frac{\rho_n}{n} \qquad \text{------2.3.2.1}$$

where ρ_n tends to zero as $n \to \infty$.

(Which is known as Asymptotic formula)

PROOF: Let f(x) be (n+1) times differentiable at $x=x_0$, then

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2} + S(x)(x-x_0)^2 \quad \text{--2.3.2.2}$$

where $\underset{x\to x_0}{\text{limit}}\ S(x) = 0$

set x = k/n in (2.3.2.2), we get

$$f\left(\frac{k}{n}\right) = f(x_0) + f'(x_0)\left(\frac{k}{n} - x_0\right) + \frac{f''(x_0)}{2}\left(\frac{k}{n} - x_0\right)^2 + S\left(\frac{k}{n}\right)\left(\frac{k}{n} - x_0\right)^2 \quad \text{----2.3.2.3}$$

Multiplying both sides of (2.3.2.3) by $\binom{n}{k} x_0^k (1-x_0)^{n-k}$ and taking the sum from k=0 to k= n, we get

$$B_n(f;x_0) = f(x_0) \sum_{k=0}^{n} \binom{n}{k} x_0^k (1-x_0)^{n-k} + \frac{f'(x_0)}{n} \sum_{k=0}^{n} (k - nx_0) P_{n,k}(x_0)$$

$$+ \frac{1}{2n^2} f''(x_0) \sum_{k=0}^{n} (k-nx_0)^2 p_{n,k}(x_0) + \sum_{k=0}^{n} S\left(\frac{k}{n}\right)\left(\frac{k}{n} - x_0\right)^2 \binom{n}{k} x_0^k (1-x_0)^{n-k}$$

(using (2.1.1) & (2.1.2))

$$B_n(f;x_0) = f(x_0) + \frac{x_0(1-x_0)}{2n} f''(x_0) + \sum_{k=0}^{n} S\left(\frac{k}{n}\right)\left(\frac{k}{n} - x_0\right)^2 \binom{n}{k} x_0^k (1-x_0)^{n-k}$$

-------- (2.3.2.4)

(using (2.1.2), (2.1.4) & 2.1.5(A))

Designate the third term in (2.3.2.4) by S. Let $\in>0$ be given, we can find n sufficiently large that $|x - x_0| < \frac{1}{n^{1/4}}$ implies

$$|S(x)| \leq \in$$

Hence,

$$|S| \le \sum_{\left|\frac{k}{n}-x_0\right|<n^{-1/4}} \left|S\left(\frac{k}{n}\right)\right| \left(\frac{k}{n}-x_0\right)^2 \binom{n}{k} x_0^k (1-x_0)^{n-k} +$$

$$\sum_{\left|\frac{k}{n}-x_0\right|\ge n^{-1/4}} |S(k/n)| \left(\frac{k}{n}-x_0\right)^2 \binom{n}{k} x_0^k (1-x_0)^{n-k}$$

or

$$|S| \le \in \sum_{k=0}^{n} \frac{1}{n^2}(k-nx_0)^2 \binom{n}{k} x_0^k (1-x_0)^{n-k} + M \sum_{\left|\frac{k}{n}-x_0\right|\ge n^{-1/4}} \binom{n}{k} x_0^k (1-x_0)^{n-k}$$

where,

$$M = \sup_{0\le x\le 1} S(x)(x-x_0)^2$$

$$|S| \le \frac{\in x_0(1-x_0)}{n} + \frac{Mc}{n^{3/2}} \qquad \text{(using (2.1.5(A))}$$

It follows from (2.3.2.4) that

$$\left| n[B_n(f;x_0) - f(x_0)] - \frac{x_0(1-x_0)}{2} f''(x_0) \right| = |nS| \le \in x_0(1-x_0) + \frac{Mc}{n^{3/2}}$$

since $\in$ is arbitrary, we get result.

Because of this result (2.3.2.1), it has been conjectured earlier that a continuous function f defined on [0,1] can satisfy

$| B_n(f;x) - f(x) | = 0(1/n)$

uniformly on some interval if f is linear on that interval.

The following theorem establishes the validity of this conjecture which was proved by K.de Leuew[13]

Theorem (2.3.3): Let f be a continuous function with domain [0,1] and let $0 \le a < b \le 1$. Then the following are equivalent.

(i) $|B_n(f;x) - f(x)| = 0(1/n)$ ----- 2.3.3.1

uniformly on every interior sub-interval of [a,b].

(ii) On each interior sub-interval of [a, b], f has bounded second derivative in the following sense.

If $a < c < d < b$, there is function h bounded and measurable in [c,d], so that if

$$h_1(x) = \int_c^x h(t)\, dt \,,\, , \; h_2(x) = \int_c^x h_1(t)\, dt \,, \qquad \text{-----2.3.3.2}$$

where $c \le x \le d$, Then f and h_2 differ by linear function on [c,d]

If (i) hold and if in addition

$$|f(x) - B_n(f;x)| = 0(1/n)$$

at almost all points of [a,b], then f is linear in [a,b]

B. Bajsanki and R. Bojanic give a simple proof of the original conjecture.

Theorem (2.3.4): If f is continuous on [0,1] and $|B_n(f;x) - f(x)| = 0(1/n)$ holds for all fixed $x \in (\alpha,\beta)$, then f is linear function on $[\alpha,\beta] \subset [0,1]$.

2.4. SIMULTANEOUS APPROXIMATION OF FUNCTIONS AND DERIVATIVES:

In contrast to other modes of approximation - in particular to Tschebyscheff or best uniform approximation.

The Bernstein polynomials yield smooth approximants. If the approximated function is differentiable, not only do we have

$B_n(f;x) \to f(x)$, but $B_n'(f;x) \to f'(x)$

A corresponding statement is true for higher derivatives.

Lemma 2.4.1: Let p $\geq$ 0 be an integer, then

$$B_{n+p}^{(p)}(f;x) = \frac{(n+p)!}{n!}\sum_{t=0}^{n} \Delta^p f\left(\frac{t}{n+p}\right)\binom{n}{t}x^t(1-x)^{n-t}$$

PROOF: Leibnitz formula is

$$(uv)^{(p)} = \sum_{j=0}^{p}\binom{p}{j}u^{(j)}v^{(p-j)} \quad \text{-----2.4.1}$$

Apply (2.4.1) in (2.1.1), we have

$$B_{n+p}^{(p)}(f;x) = \sum_{k=0}^{n+p} f\left(\frac{k}{n+p}\right)\binom{n+p}{k}\sum_{j=0}^{p}\binom{p}{j}(x^k)^{(j)}\{(1-x)^{n+p-k}\}^{(p-j)} \text{--2.4.2}$$

Now, we have

$$(x^k)^{(j)} = \frac{k!x^{k-j}}{(k-j)!}, \qquad k-j \geq 0$$

and

$$\left[(1-x)^{n+p-k}\right]^{(p-j)} = (-1)^{(p-j)}(n+p-k)!\frac{(1-x)^{n+j-k}}{(n+j-k)!}, \qquad k-j \leq n$$

Therefore (2.4.2) becomes

$$B_{n+p}^{(p)}(f;x)=\sum_{k=0}^{n+p}\sum_{\substack{j=0\\0\le k-j\le n}}^{p} f\left(\frac{k}{n+p}\right)\frac{(n+p)!}{k!(n+p-k)!}$$

$$\binom{p}{j}\frac{k!x^{k-j}}{(k-j)!}(-1)^{p-j}\frac{(n+p-k)!}{(n+j-k)}(1-x)^{n+j-k}$$

$$=\sum_{k=0}^{n+p}\sum_{\substack{j=0\\0\le k-j\le n}}^{p} f\left(\frac{k}{n+p}\right)\frac{(n+p)!}{(k-j)!(n+j-k)!}\binom{p}{j}(-1)^{p-j}x^{k-j}(1-x)^{n+j-k}$$

Set k-j= t and 0≤ t ≤ n, j=0, 1,2, ------,p

$$B_{n+p}^{(p)}(f;x)=(n+p)!\sum_{t=0}^{n}\frac{x^t(1-x)^{n-t}}{t!(n-t)!}\sum_{j=0}^{p}(-1)^{p-j}\binom{p}{j}f\left(\frac{t+j}{n+p}\right) \text{-2.4.3}$$

Therefore, we get result

$$B_{n+p}^{(p)}(f;x)=\frac{(n+p)!}{n!}\sum_{t=0}^{n}\Delta^p f\left(\frac{t}{n+p}\right)\binom{n}{t}x^t(1-x)^{n-t}$$

where,

$\Delta^0 y_k = y_k$

$\Delta^1 y_k = y_{k+1}-y_k$

$\Delta^2 y_k = y_{k+2}-2y_{k+1}+y_k$

$\Delta^3 y_k = y_{k+3}-3y_{k+2}+3y_{k+1}-y_k$

In general

$$\Delta^n y_k=\sum_{r=0}^{n}(-1)^{n-r}\binom{n}{r}y_{k+r}$$

Theorem 2.4.2: Let f(x) $\in C^p[0,1]$, then

$$\lim_{n\to\infty} it\ B_n^{(p)}(f;x) = f^{(p)}(x), \qquad \text{uniformly on } [0,1]$$

PROOF: (See Davis[06], on page 113-114)

CHAPTER – 3

APPROXIMATION OF FUNCTION BY GENERALIZED BERNSTEIN POLYNOMIAL

CHAPTER – 3

APPROXIMATION OF FUNCTION BY GENERALIZED BERNSTEIN POLYNOMIAL

3.1. INTRODUCTION:

If $f(x_1, x_2, \text{--------}, x_N)$ is continuous on hypercube C: $0 \le x_j \le 1$, j = 1,2, ------, N, then the generalized Bernstein polynomial

$$B(f; x_1, x_2 \text{-----}, x_N) = \sum_{k_1=0}^{n_1} \text{-----} \sum_{k_N=0}^{n_N} \binom{n_1}{k_1}\binom{n_2}{k_2} \text{------} \binom{n_N}{k_N} \times$$

$$f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}, \text{-----} \frac{k_N}{n_N}\right) x_1^{k_1}(1-x_1)^{n_1-k_1} \text{------} x_N^{k_N}(1-x_N)^{n_N-k_N} \qquad \text{--------} 3.1.1$$

Converges uniformly in C is to f as $\min_j n_j \to \infty$

The Weierstrass Theorem has been generalized in many different directions. We shall look at generalization to functions of N real variables. If a real function of N real variables is continuous on a closed bounded set of R_N; it may be approximated uniformly by polynomials in the N- variables. There are many proofs of this fact, one proof – an extension of **Theorem 2.1.** makes use of generalized Bernstein polynomial (3.1.1)

In last chapter, we have seen that if $f'(x) \in \text{Lip}(1)$, then

$$|(f(x) - B_n(f;x)| \le C \left(\frac{1}{n}\right)$$

But this estimate can not be improved the order of approximation by assuming the existence of the second or higher derivative of f(x), then

$|f(x) - B_n(f;x)| = 0\left(\frac{1}{n}\right)$ can never be true except if f(x) is a linear function.

Butzer [2] shows that if we take the linear combination of $B_{n,}(f;x)$ in place of $B_n(f;x)$, then we obtained a better degree of approximation in comparison of $B_n(f;x)$ under certain condition.

Polynomial approaching f(x) more closely than the Bernstein Polynomials, but of a different type from those considered here, were also considered by Bernstein [3].

$$Q_n\,(f;x) = \sum_{k=0}^{n}[f(x/n) - \frac{x(1-x)}{2}f''\left(\frac{x}{n}\right)]p_{nk}(x)$$

Where,

$$p_{n,k}\,(x) = \binom{n}{k}x^k(1-x)^{n-k}$$

Theorem 3.1: If $|f(x)| \leq M$ and if $f^{(4)}$ (x) exists, it can be shown that

$$\lim_{n\to\infty} it\; n^2\,[Q_n\,(f;x) - f(x)] = \frac{x(1-x)(1-2x)}{6}f^{(3)}{}_{(x)} - \frac{[x(1-x)]^2}{8}f^{(4)}{}_{(x)}$$

3.2. APPROXIMATION OF FUNCTION WITH GENERALIZED BERNSTEIN POLYNOMIAL:

Let f be a function continuous on [0,1] If n is an integer greater than 1, Kerel de Leuew [13] defines the n^{th} modified Bernstein Polynomial $A_{n,r}$ of f, defined by

$$A_{n,r} = \sum_{r=1}^{n-1} f(n,r) p_{n,k}(x) \qquad \text{-----3.2 (a)}$$

Where,

$$f(n,r) = n \int_{-1/2n}^{+1/2n} f\left(\frac{r}{n} + t\right) dt$$

He has proved the following results.

Theorem 3.2.1: Let f be a continuous function defined on [0,1], Let $0<c<d<1$ and suppose that for $x \in [c,d]$ and all h

$$|f(x+h) + f(x-h) - 2f(x)| \leq M|h|$$

then $$|A_n(f;x) - B_n(f;x)| = 0\left(\frac{1}{n}\right)$$

uniformly on all interior subintervals of [c,d].

Theorem 3.2.2: Let f be a continuous function defined on [0,1], Let $0 \leq a < b \leq 1$, then the following are equivalent.

(i) $|B_n(f;x) - f(x)| = 0\left(\frac{1}{n}\right)$ uniformly on all interior subinterval of [a,b]

(ii) $|A_n(f;x) - f(x)| = 0\left(\frac{1}{n}\right)$ uniformly on all interior subinterval of [a,b]

L.C Hsu [10] introduces a kind of polynomials of the form.

For $f(x) \in [0,1]$, he defines

$$P_n(f;x) = \frac{1}{\sqrt{n\pi}}\sum_{k=0}^{n} f(k/n)\left[1-\left(\frac{k}{n}-x\right)^2\right]^n \qquad \text{----- 3.2 (b)}$$

The Polynomial $P_n(f;x)$ is just as simple as $B_n(f;x)$.

Moreover, it has in common with $B_n(f;x)$, pecularity of using only the values f(k/n) (k = 0,1,2,... n) in its construction.

For $f(x) \in L^p(0,1)$ with $p \geq 1$, we define

$$S_n(f;x) = \sqrt{\frac{n}{\pi}}\sum_{k=1}^{n}\left[1-\left(\frac{k}{n}-x\right)^2\right]^n \int_{\frac{k-1}{n}}^{k/n} f(t)\,dt \qquad \text{---- 3.2 (c)}$$

This is again a polynomial in x

L.C. Hsu [10] establish a pair of theorem as follows.

Theorem 3.2.3: For a continuous function f(x) defined on [0,1] the relation

$$\operatorname*{limit}_{n\to\infty} P_n(f;x) = f(x) \qquad \text{------3.2 (d)}$$

holds uniformly on $n \leq x \leq 1-n$, where n is arbitrary small fixed number with $o < n < ½$.

Theorem 3.2.4: The relation

$$\lim_{n\to\infty} it \int_0^1 | S_n(f;x) - f(x) |^p dx = 0 \qquad \text{-----3.2 (e)}$$

is true for all f(x) $\in L^p$ (0,1), p$\geq$ 1.

L.C. Hsu [10] also defines another type of polynomial.

$$T_n(f;x) = \frac{1}{n}\sum_{k=0}^{n} f\left(\frac{k}{n^{3/4}}\right)\left[1-\left(\frac{k}{n}-\frac{x}{n^{3/4}}\right)^2\right]^n \qquad \text{-----3.2 (f)}$$

This is again a polynomial in x.

He has proved another result, which satisfies as:

Theorem 3.2.5: If f(x) is a real and continuous function defined on [0,1], thus the relation.

$\lim_{n\to\infty} it\, T_n(f;x)$ = f(x) holds uniformly

Later T.A. Azlerov and H.Mansurov [01] has proved the following results.

Theorem 3.2.6: If f(x) $\in$ C[0,1], thus

$$|P_n(f;x) - f(x)| \leq 3/2\ \omega\left(\frac{1}{\sqrt{n}}\right) + \frac{c}{n}$$

0 < x < 1, n $\geq$ 1 where ω (h) be the modulus of continuity of f(x) and c is a constant independent of n.

The order of approximation of theorem (3.2.3) established by theorem (3.2.6).

From the above results, we conclude that the polynomials 3.2(a), 3.2(b), 3.2 (c) and 3.2 (f) approximate the function f(x) more closely than $B_n(f;x)$.

Theorem 3.2.7: If f(x) $\in C[0,1]$ and f has a finite second derivative at x, o < x < 1, thus

$$P_n(f;x) = f(x) + \left[\frac{0.5f''(x) - 0.75f(x)}{(2n) + 0(1/n)}\right] \quad (n \to \infty) \qquad \text{-----3.2(g)}$$

To obtain the more accuracy of degree of approximation of function f(x) defined on [0,1] in comparison of the polynomial 3.2(a) 3.2(b), 3.2(c) and 3.2(f).

T.A. Azlarov and H. Mansurov [01] defines the polynomials.

$$P_{n,\alpha}(f;x) = \frac{1}{\sigma_\alpha n^\beta} \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\left[1 - \left(\frac{k}{n} - x\right)^\alpha\right]^{n^\gamma}$$

where,

$$\alpha \geq 1, \quad \beta = \frac{\alpha}{\alpha+2}, \quad \gamma = \frac{2\alpha}{\alpha+2}, \quad \sigma_\alpha = \left(\frac{2}{\alpha}\right)^{1/\alpha}$$

they prove the several results.

Theorem 3.2.8: If f(x) is a continuous function defined on [0,1], then relation

$$\operatorname{limit}_{n\to\infty} \sup_{0<x<1} \left|P_{n,\alpha}(f;x) - f(x)\right| = 0$$

holds uniformly on the segment [0,1]. The order of approximation of theorem 3.2.8 established by theorem:

Theorem 3.2.9: If f(x) $\in$C[0,1] and ω(f;h) be the modulus of continuity of f(x), then

$$|P_{n,\alpha}(f;x) - f(x)| \le k_1(\alpha) \quad \omega\left(f, \frac{1}{n^{1-\beta}}\right) + \frac{k_2}{n^{2\beta}} + \frac{k_3}{n^{p}}\omega\left(f, \frac{1}{n^{1-\beta}}\right)$$

where

$k_1(\alpha) = 1 + \frac{2}{\sigma_\alpha}\int_1^\infty y e^{-y^\alpha} dy$ and k_2, k_3 are constant not

depend on n

Theorem 3.2.10: Let $\omega_0(\delta)$ be a fixed modulus of continuity and let C be the class of functions f(x) for which

$$\omega(f;\delta) \le \omega\ (0;\delta)\ ,\ o \le \delta \le 1$$

and if

$$E_{n,\alpha}(x, C_\omega) = \sup_{f\in C_\omega} \left\| P_{n,\alpha}(f;x) - f(x) \right\|$$

then for all $n \ge 1$

$$E_{n,\alpha}(x, C_\omega) = \frac{1}{\sigma_\alpha n^\beta}\sum_{k=0}^{n}\omega_0 |(k/n - x)|^n [1-(k/n-x)^\alpha]^{n^r} + 0\left(\frac{1}{n^{2\beta}}\right)$$

From the relation 3.2(d), 3.2(g) and for f$\in C^2$, where C^2 be the class of functions having second derivatives. Azlarov proved the following asymptotic formula.

$P_{n,2}$ (f;x) =f(x)+ $\dfrac{f''(x)-1.5f(x)}{4n}+0(1/n)$

If $\overline{P}_{n,2}(f;x)=\dfrac{8n}{8n-3}P_{n,2}(f;x),$ then

$$\overline{P}_{n,2}(f;x)=f(x)+\frac{f''(x)}{n}+0(1/n)$$

and in general case, he consider

$$\overline{P}_{n,\alpha}(f;x)=\frac{P_{n,\alpha}(f;x)}{1-\dfrac{\alpha+1}{2\alpha^2}.n^{-2\beta}}$$

Then he proved the following result

Theorem 3.2.11: Let $f\in C^2$ and $\alpha>\dfrac{(\sqrt{17}-1)}{2}$

then,

$$\overline{P}_{n,\alpha}(f;x)=f(x)+\frac{\Gamma 3/\alpha}{2\,\Gamma 1/\alpha}\frac{f''(x)}{n^{2-2\beta}}+0\left(\frac{1}{n^{2-2\beta}}\right)$$

3.3. GENERALIZATION OF BERNSTEIN POLYNOMIAL:

G.H. Kirov [12] investigates a generalization of the Bernstein Polynomial that if f: [0,1] → R, $f\in c^r$, $r\in\{0,1,2 \text{ --- }\} = N$, then for every $n\in N$ and $f\in C^r$, the polynomials

$$B_{n,r}(f;x) = \sum_{k=0}^{n}\sum_{i=0}^{r}\frac{1}{i!}f^{(i)}\left(\frac{k}{n}\right)\left(x-\frac{k}{n}\right)^i\binom{n}{k}x^k(1-x)^{n-k}$$

are introduced.

The polynomial $B_{n,r}$ (f;x) are a generalizations of the classical Bernstein Polynomials.

G.H. Kirov [12] proves two theorems which are natural generalizations of the classical results due to popoviciu and Voronovskaya.

Sofiya and Ostrovska [23] study some properties of the family of positive linear operators defined on C[0,1], by

$$B_{n,q}\ (f)\ (x) = \sum_{k=0}^{n} f\left(\frac{n}{k_q} x^k \prod_{S=0}^{n-k-1}(1-q^s x)\right)$$

$$(0 \leq x \leq 1)$$

for every $n \geq 1$ and $q \in \Re, \quad q > o,$

Here,

$$[n]_q := \left\{\begin{matrix} 1+q+------+q^{n-1}, & n \geq 1 \\ 0 & n=0 \end{matrix}\right\}$$

$$[n]_q! := \left\{\begin{matrix} [1]_q \quad [2]_q -----[n]_q & ,n \geq 1 \\ 1 & n=0 \end{matrix}\right\}$$

and for $o \leq k \leq n$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$$

These operators were first introduced and studied by G.M. Phillips. Sofiya, Ostrovska [23] investigate the behaviour of $B_{n,q}$ as $n \to \infty$ for any fixed q and they show that for every $f \in (C[0,1])$, there exists the limit.

$$\underset{n\to\infty}{\text{limit}}\ B_{n,q}(f) = B_{\infty,q}(f),\ \text{uniformly on } C([0,1])$$

Moreover, above limit is uniform, when q ranges in a compact subinterval of [0,1].

Sofiya, ostrovska [23] also study some properties of the limit operator $B_{\infty,q}$ and among other thing, they show that for every $f \in C\,[0,1]$

$$\underset{q\to 1^-}{\text{limit}}\ B_{\infty,q}(f) = f,\ \text{uniformly on } C([0,1])$$

CHAPTER-4

LIPSCHITZ CONSTANTS FOR THE BERNSTEIN POLYNOMIALS

CHAPTER-4

LIPSCHITZ CONSTANTS FOR THE BERNSTEIN POLYNOMIALS

4.1. INTRODUCTION:

We shall assume that the given function f is Lipschitz continuous of order μ , $0 < \mu \leq 1$, on [0,1]. that is, there exists a constant $A \geq 0$ such that for every pair of points $x_1, x_2 \in [0,1]$, we have

$$|f(x_1)-f(x_2)| \leq A\ |x_1 - x_2|^{\mu} \qquad \text{------4.1.1}$$

obviously, if f is differentiable on [0, 1], then f satisfies inequality (4.1.1).

The interesting thing about this chapter is that each of the Bernstein Polynomials $B_n(f)$, for n =1, 2, 3, ----has the same Lipschitz constant as the given function f when considered as being in the class of functions Lip. μ.

4.2. BERNSTEIN INEQUALITY

The n^{th} ($n \geq 1$) Bernstein polynomial of f is defined by

$$B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \qquad \text{------4.2.1}$$

Obviously, $B_n(f)$ is a polynomial of degree $\leq$ n and its importance in the approximation theory arises from the fact that

$$\operatorname*{limit}_{n\to\infty} B_n(f;x) = f(x) \text{ uniformly on } [0,1] \qquad \text{-----4.2.2}$$

Davis[06] gives elementary properties of these polynomials one of the outstanding properties of these polynomials is that they mimic the behaviour of given function f to a remarkable degree.

Thus if, in addition to being continuous on [0,1], f is convex, then $B_n(f)$ is also convex.

consider

$$(1-x)^{-n}[B_{n-1}(f;x)-B_n(f;x)]$$

put $x=\dfrac{t}{1+t}$

$$=\left(1-\frac{t}{1+t}\right)^{-n}\left[\sum_{k=0}^{n-1}\binom{n-1}{k}f\left(\frac{k}{n-1}\right)\left(\frac{t}{1+t}\right)^{k}\left(1-\frac{t}{1+t}\right)^{n-1-k}-\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)\left(\frac{t}{1+t}\right)^{k}\left(1-\frac{t}{1+t}\right)^{n-k}\right]$$

$$=(1+t)^{n}\left[\sum_{k=0}^{n-1}\binom{n-1}{k}f\left(\frac{k}{n-1}\right)\left(\frac{t}{1+t}\right)^{k}\left(\frac{1}{1+t}\right)^{n-1-k}-\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)\left(\frac{t}{1+t}\right)^{k}\left(\frac{1}{1+t}\right)^{n-k}\right]$$

$$=\sum_{k=0}^{n-1}\binom{n-1}{k}f\left(\frac{k}{n-1}\right)\frac{t^{k}}{(1+t)^{k}}\left(\frac{1}{(1+t)^{n-1-k}}\right)(1+t)^{n}-\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)\frac{t^{k}}{(1+t)^{k}}\frac{1}{(1+t)^{n-k}}(1+t)^{n}$$

$$=(1+t)\sum_{k=0}^{n-1}\binom{n-1}{k}f\left(\frac{k}{n-1}\right)t^{k}-\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)t^{k}$$

$$=(1+t)\left[f(0)+\sum_{k=1}^{n-1}\binom{n-1}{k}f\left(\frac{k}{n-1}\right)t^{k}\right]-\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)t^{k}$$

$$=f(0)+\sum_{k=1}^{n-1}\binom{n-1}{k}f\left(\frac{k}{n-1}\right)t^{k}+t.f(0)+t\sum_{k=1}^{n-1}\binom{n-1}{k}f\left(\frac{k}{n-1}\right)t^{k}-\sum_{k=0}^{n}\binom{n}{k}f\left(\frac{k}{n}\right)t^{k}$$

$$= f(0)+\sum_{k=1}^{n-1}\binom{n-1}{k} f\left(\frac{k}{n-1}\right)t^k + t\left[\ f(0)+\sum_{k=1}^{n-2}\binom{n-1}{k} f\left(\frac{k}{n-1}\right)t^k\ \right]$$

$$+ f(1)t^n - \sum_{k=0}^{n}\binom{n}{k} f\left(\frac{k}{n}\right)t^k$$

$$= f(0)+\sum_{k=1}^{n-1} f\left(\frac{k}{n-1}\right)\binom{n-1}{k}t^k + \sum_{k=1}^{n-1}\binom{n-1}{k-1} f\left(\frac{k-1}{n-1}\right)t^k + f(1)t^n$$

$$-\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right)\binom{n}{k}t^k - f(0) - f(1)t^n$$

$$= \sum_{k=1}^{n-1}\left[\binom{n-1}{k} f\left(\frac{k}{n-1}\right) + f\left(\frac{k-1}{n-1}\right)\binom{n-1}{k-1} - \binom{n}{k} f\left(\frac{k}{n}\right)\ \right]t^k \qquad \text{-----4.2.3.}$$

Let

$$C_k = \frac{(n-1)!}{(k-1)!(n-k-1)!}\left\{\frac{1}{k} f\left(\frac{k}{n-1}\right)+\frac{1}{n-k} f\left(\frac{k-1}{n-1}\right)-\frac{n}{k(n-k)} f\left(\frac{k}{n}\right)\right\}$$

-----4.2.4.

4.2.3 equals to

$$= \sum_{k=1}^{n-1} C_k\, t^k$$

Now,

$\frac{k-1}{n-1} < \frac{k}{n} < \frac{k}{n-1}$, and since f is convex, then

$$\left\{\frac{1}{k} f\left(\frac{k}{n-1}\right)+\frac{1}{(n-k)} f\left(\frac{k-1}{n-1}\right)-\frac{n}{k(n-k)} f\left(\frac{k}{n}\right)\right\} \geq 0$$

Therefore,

$$\sum_{k=1}^{n-1} C_k\, t^k \geq 0$$

Therefore, we have an exact formulation of this leading reasoning

$B_{n-1}(f;x) \geq B_n(f;x) \geq f(x)$, $o < x < 1$ -------4.2.5

Corollary : If f is linear in each of the intervals $\left[\frac{j-1}{n-1}, \frac{j}{n-1}\right]$, then all the C_j are O and hence

$$B_{n-1} \equiv B_n$$

Conversely, if $B_{n-1} \equiv B_n$, then all the C_j are O and since $f \in C[0,1]$ and is convex, (4.2.4) implies that f is linear in each interval.

4.3. LIPSCHITZ CONSTANT FOR BERNSTEIN POLYNOMIAL

As a consequence of result (4.2.5), we have

$$B_n(x^{\mu}; h) \leq h^{\mu} \quad , \quad 0 \leq h \leq 1 \qquad \text{------4.3.1}$$

Let $x_1 \leq x_2$ be any two points of [0,1], then.

$$B_n(f;x_2) = \sum_{j=0}^{n} \binom{n}{j} (1-x_2)^{n-j} f\left(\frac{j}{n}\right)[x_1 + (x_2 - x_1)]^j$$

$$= \sum_{j=0}^{n} \binom{n}{j} (1-x_2)^{n-j} f\left(\frac{j}{n}\right)\left\{\sum_{k=0}^{j} \binom{j}{k} x_1^{\,k} (x_2 - x_1)^{j-k}\right\}$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{n!\, x_1^{\,k} (x_2 - x_1)^{j-k} (1-x_2)^{n-j}}{k!\,(j-k)!\,(n-j)!} f(j/n)$$

on inverting the order of summation and writing $k+\ell = j$, then

$$B_n(f;x_2) = \sum_{k=0}^{n} \sum_{\ell=0}^{n-k} \frac{n!}{k!\,\ell!\,(n-k-\ell)!} x_1^{\,k} (x_2 - x_1)^{\ell} \times (1-x_2)^{n-k-\ell} f\left(\frac{k+\ell}{n}\right) \qquad \text{---- 4.3.2}$$

We now construct a similar double sum for $B_n(f;x_1)$

$$B_n(f;x_1) = \sum_{K=0}^{n} \binom{n}{k} x_1^{\,k} f\left(\frac{k}{n}\right)[(x_2 - x_1) + (1-x_2)]^{n-k}$$

$$= \sum_{K=0}^{n} \binom{n}{k} x_1^{\,k} f\left(\frac{k}{n}\right) \left\{ \sum_{\ell=0}^{n-k} \binom{n-k}{\ell} \times (x_2 - x_1)^{\ell} (1 - x_2)^{n-k-\ell} \right\}$$

$$= \sum_{K=0}^{n} \sum_{\ell=0}^{n-k} \frac{n!}{k!\,\ell!\,(n-k-\ell)!} x_1^{\,k} (x_2 - x_1)^{\ell} \times (1 - x_2)^{n-k-\ell} f\left(\frac{k}{n}\right) \text{------ } 4.3.3$$

on subtracting (4.3.3) from (4.3.2), we have

$$|B_n(f;x_2) - B_n(f;x_1)| = \left| \sum_{k=0}^{n} \sum_{\ell=0}^{n-k} \frac{n!}{k!\,\ell!(n-k-\ell)!} x_1^{\,k} (x_2 - x_1)^{\ell} (1 - x_2)^{n-k-\ell} \times \left\{ f\left(\frac{k+\ell}{n}\right) - f\left(\frac{k}{n}\right) \right\} \right|$$

$$\leq A \sum_{K=0}^{n} \sum_{\ell=0}^{n-k} \frac{n!}{k!\,\ell!\,(n-k-\ell)!} x_1^{\,k} (x_2 - x_1)^{\ell} (1 - x_2)^{n-k-\ell} \left(\frac{\ell}{n}\right)^{\mu} \text{-----[from 4.1.1)}$$

$$= A \sum_{\ell=0}^{n} \frac{(x_2 - x_1)^{\ell}\, n!}{\ell!\,(n-\ell)!} \left(\frac{\ell}{n}\right)^{\mu} \left\{ \sum_{k=0}^{n-\ell} \binom{n-\ell}{k} x_1^{\,k} (1 - x_2)^{n-k-\ell} \right\}$$

$$= A \sum_{\ell=0}^{n} \binom{n}{\ell} (x_2 - x_1)^{\ell} \left(\frac{\ell}{n}\right)^{\mu} (x_1 + 1 - x_2)^{n-\ell}$$

$= A\ B_n\ (x^{\mu};\ x_2 - x_1)$ [from 4.2.1]
$\leq A\ (x_2 - x_1)^{\mu}$ [from 4.3.1]

By choosing f(x) to be $A\ x^{\mu}$ and the points x_1, x_2 to be 0,1 respectively, we see that the constant A cannot be diminished for any value of n. **Bloom and Elliott [04]** showed that it was true when $\mu = 1$ and for $\mu \neq 1$ showed that $B_n(f) \in Lip_A(\mu/4)$ Dr. Dickmeis stated that the result was true as a consequence of the Peetre K. theory of interpolation between Banach spaces.

The following theorem gives an exact formulation of this leading reasoning

Theorem 4.3. If $f \in Lip._A(\mu)$, then for all $n \geq 1$,

$$\mathbf{B_n(f) \in Lip._A(\mu) \text{ also.}}$$

4.4. LIPSCHITZ CONSTANT OF THE BERNSTEIN POLYNOMIAL ON A RECTANGLE

Let f(x,y) be a real valued function defined on the rectangle

$$R: = \{(x,y): 0 \leq x \leq 1, \quad 0 \leq y \leq 1\}, \textit{ and}$$

B_{mn} (f;x,y) be the corresponding Bernstein polynomial of order (m,n).

Cheng, Fa Lai[05] proved that if f(x,y) is Lipschitz continuous i.e. $f(x,y) \in Lip_A\alpha$, then for all positive integers m and n, we have

$$B_{mn}(f;x,y) \in Lip_B\alpha$$

Here $B = A\sqrt{2^\alpha}$ and, in a sense, constant B is the best. At the end, the above result is generalized to the higher dimensional case.

CHAPTER-5

BERNSTEIN POLYNOMIAL ON A SIMPLEX

CHAPTER-5

BERNSTEIN POLYNOMIAL ON A SIMPLEX

5.0: INTRODUCTION: Ping [20],Proved the theorems for functions and give an estimation of the error for the difference of two consecutive Bernstein Polynomial as defined on a simplex, Xiong, Jing & Yang [28] introduces derivatives and smoothness of functions on a simplex, In this chapter, we shall show their result.

5.1: ON LINEAR COMBINATIONS OF DERIVATIVES:

Let T be a simplex in R^d, $d \in N$. First **Derriennic [8]** introduces an elliptic differential operator L^r on $C^{2r}(T)$, $r \in N$, where L can be considered as a multivariate generalization of the Legendre differential operator $Lf = \left(\frac{d}{dx}\right) x\,(1-x)\, f'(x)$ and where $L^r f = L(L^{r-1} f)$.

Then a new polynomial operator $M_n^r(f)$ consisting of linear combinations of the introduced multivariate "Legendre" derivatives of Bernstein Type Polynomials is constructed. Using the closed connection between the Bernstein type operator M_n and the differential operator L,

Derriennic [08] proves, besides other results, that for

$f \in L^p(T)$, $1 \leq p\ \infty$,

$|| M_n^{r-1} f - f ||_p \leq C(n^{-r/2} || f ||_p + \omega_{2r}^p(f, n^{-1/2}))$

(C being a constt independent of n and f).

Corresponding inverse theorem are also given.

It should be mentioned that the operator M_n^r has analogue approximation properties analogous to those of a Bernstein type operator (Introduced by Ditzian and Ivanov). In the one dimensional case, as a further consequence of the derived results, **Derriennic[08]** obtains that in the one dimensional case d=1

$\| M_n^{r-1} f - f \|_p = 0(n^{-\alpha})$ is equivalent to

$E_n^p(f) = 0(n^{-2\alpha})$, where $1 \leq p \leq \infty$, $\alpha < r$ and $E_n^p(f)$ denotes as usual, the minimum deviation of f w.r.t. the set of algebraic polynomial of degree less than or equal to n.

5.2. DERIVATIVES AND SMOOTHNESS OF FUNCTIONS

Let $T = \{(x,y) \in \Re^2 : x, y \geq 0 \text{ and } x+y \leq 1\}$. for $f \in C(T)$

Let B_n (f; x, y) be the n^{th} Bernstein polynomial of f(x,y) and let

$$\Delta_{h_1} f(x,y) = f(x+h, y) - f(x,y)$$

$$\Delta^2{}_{h_1} f(x,y) = \Delta_{h_1} (\Delta_{h_1} f(x,y))$$

$$\Delta_{h_2} f(x,y) = f(x, y+h) - f(x,y) \text{ and}$$

$$\Delta^2{}_{h_2} f(x,y) = \Delta_{h_2} (\Delta_{h_2} f(x,y))$$

Denote $\omega_r^{(i)}(f,n) = \sup_{0<h\leq\eta} \left|\Delta_{h_i}^r f(x,y)\right|$

where i,r = 1,2 --------------

Let $\tau(n,x) = \min (n^2, n(1-y) / x(1-x-y))$

And also let

$$\tau(n,y) = \min\,(n^2,\ n(1-x)/y(1-x-y)),$$

Xiong, Jing & Yang [28], proves that for $f \in C(T)$,

$\eta > 0$ and $\omega_1^{(i)}\,(f,t) \leq Mt^n$, where $i = 1,2,$

one has

(a) If $0 < \alpha \leq 1$, then

$$|(\partial/\partial x)\ B_n(f\ ;\ x,\ y)| \leq M_1 \tau(n,x)^{(1-\alpha)/2}$$

iff $\quad \omega_1^{(1)}(f,h) \leq M_2 h^{\alpha};$

and

$$|(\partial/\partial y)\ B_n(f\ ;\ x,\ y)| \leq M_3 \tau(n,y)^{(1-\alpha)/2}$$

iff $\quad \omega_1^{(2)}\,(f,h) \leq M_4 h^{\alpha};$

(b) If $0 < \alpha \leq 2$, then

$$|(\partial^2/\partial x^2)\ B_n(f\ ;\ x,\ y)| \leq M_5 \tau(n,x)^{(2-\alpha)/2}$$

iff $\quad \omega_2^{(1)}\,(f,h) \leq M_6 h^{\alpha};$

and

$$|(\partial^2/\partial y^2)\ B_n(f\ ;\ x,\ y)| \leq M_7 \tau(n,y)^{(2-\alpha)/2}$$

iff $\quad \omega_2^{(2)}\,(f,h) \leq M_8 h^{\alpha};$

This is a two-dimensional version of a result of Ditzian on Bernstein Polynomial $B_n\,(f;x)$ of $f(x)$, where $f \in C[0,1]$

5.3. ON THE MAXIMUM PRINCIPLE

Reviewer & Zhang [22] proved the so-called converse theorems on convexity of Bernstein polynomial on a triangle.

In extending these results to the multivariate setting,

Dahman & Micchelli [07] proved among other things the following.

Theorem 5.3.1: Suppose f is a continuous function on S_m such that for the Bernstein polynomials $B_n f$, the following holds:

$$B_n f(u) \geq f(u), \quad u \in S_m, \quad n \in N$$

then f achieves its maximum on ∂S_m (the boundary of the m-dimensional simplex).

Sauer & Thomas [26] gives a very simple proof for a result which implies Theorem (5.3.1). By a repeated use of this theorem.

Theorem 5.3.2: Suppose $f \in C(S_m)$ is such that

$$B_n f \geq f$$

then f achieves its maximum at one of the vertices of S_m.

5.4. ON ASYMPTOTIC EXPANSION:

By **Lorentz[16]**, it can found that if $f \in C^{2S}[0,1]$ then following asymptotic expansion holds:

$$B_n(f;x) = f(x) + \sum_{k=1}^{2S} \frac{1}{k!} n^{-k} f^{(k)}(x) T_{n,k}(x) + \epsilon_n n^{-S}$$

where, $\epsilon_n \to 0$ as $n \to \infty$ and

$$T_{n,k}(x) = \sum_{i=0}^{n}(i-nx)^k\binom{n}{i}x^i(1-x)^{n-i}$$

Ping[20], proves the following theorems for functions

Theorem 4.1: For $f \in C^S(\sigma)$, we have,

$$\left|B_n(f,\lambda)-\sum_{k=0}^{S}\frac{1}{n^k}B_{n+1}(F_kf,\lambda)\right| \le \frac{2^S\omega_S\left(2/n\right)}{n^S S!}B_{n+1}(g_s,\lambda)$$

where F_k is the non-linear partial differential operator of order k defined by

$$F_k\, f(\lambda) := \sum_{|a|=k}\left(\frac{1}{a!}\right)\sum_{i=0}^{m}\lambda_i(\lambda-e^i)^a D_a f(\lambda)$$

and g_s is the polynomial

$$g_S(\lambda) = \sum_{i=0}^{m}\lambda_i(1-\lambda_i)^s$$

Theorem 5.4.2: If $f \in C^{2S}(\sigma)$, then

$$B_nf - B_{n+1}f = \sum_{k=1}^{S}\left(\frac{1}{n^{k+1}}\right)T_K f + \in_n n^{-S-1},$$

$$B_nf = \sum_{k=0}^{S}\left(\frac{1}{n^k}\right)A_K f + \delta_n n^{-S}$$

Where $\in_n$ and δ_n tend to zero uniformly as n goes to infinity.

Theorem 5.4.3: If $f \in C^{S+2}(\sigma)$, then

$$\lim_{n\to\infty} it\ n^{S+1}\Delta_n^s B_n(f) = S!\,F_2 f$$

Ping[20], gives an estimation of the error for the difference of two consecutive Bernstein polynomials defined on a simplex, and an asymptotic expansion of Bernstein polynomial.

Dahmen and Micchelli [07] have generalized the asymptotic expansion to Bernstein polynomials defined on a simplex.

Let

$x:=(x_1, x_2, -----, x_d) \in R^d$ and

$k:=(k_1, k_2, -----, k_d) \in N_0^d$

Let

$|x| : = x_1 + x_2 + ----- + x_d$

$|k| : = k_1 + k_2 + ----- + k_d$

$x^k : = x_1^{k_1}, x_2^{k_2} ------ x_d^{k_d}$ and

$k! : = k_1! ----- k_d!$

Let

$\alpha x : = (\alpha x_1, -----, \alpha x_d)$ for $\alpha \in R$.

$x-y : = (x_1-y_1, -----, x_d-y_d)$ and

$x \geq y$ mean $x_i \geq y_i$ for $i = 1, 2, -----, d$

Let $S \subset R^d$ $(d \in N)$ denote a simplex defined by

$$S : = \{x \in R^d : x \geq 0,\ 1-|x| \geq 0\}$$

Ulrich & Mircea [27] consider a multivariate extension of the celebrated Bernstein polynomials for $f: S \rightarrow R$ and $n \in N_0$ of the form

$$B_n(f;x) = \sum_{|k|\le n} p_{n,k}(x) f\left(\frac{k}{n}\right), \quad x\in S$$

Where

$$p_{n,k}(x) := \frac{n(n-1)------(n-|k|+1)}{k!} x^k (1-|x|^{n-|k|}), \quad |k|>0$$

and

$$p_{n,0}(x) := (1-|x|)^n$$

In the case d=1, the polynomials $B_n(f;x)$ coincide with the univariate Bernstein polynomials

Let $x\in S$ be fixed. **Ulrich & Mircea[27]** establish the following point wise asymptotic expansion:

$$B_n(f;x) = f(x) + \sum_{k=1}^{q} n^{-k} \sum_{k<|S|\le 2k} \frac{1}{S!}\left(\frac{\partial^{|S|}}{\partial x_1^{S_1} ----\partial x_d^{S_d}} f(x)\right)$$

$$\times \sum_{m\le S} a(k,s,m) x^{S-m} + 0(n^{-q})$$

$$n\to\infty \qquad ----5.4.1$$

For f having continuous partial derivatives of order $\le 2q$ the number a(k,S,m) is explicitly given.

In the particular case, q=1, above asymptotic equality (5.4.1) gives.

$$\lim_{n\to\infty} n[B_n(f;x)-f(x)=\frac{1}{2}\left(\sum_{j=1}^{d} x_i(1-x_i)\frac{\partial^2 f(x)}{\partial x_i^2}-2\sum_{1\le i<j\le d} x_i x_j \frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right)$$

Which is a multivariate extension of E.V. Voronovskaya's asymptotic relation.

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